

CSC 427: Data Structures and Algorithm Analysis

Fall 2008

Dynamic programming

- top-down vs. bottom-up
- divide & conquer vs. dynamic programming
- examples: Fibonacci sequence, binomial coefficient
- caching
- example: making change
- problem-solving approaches summary

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Divide and conquer

divide and conquer is a *top-down* approach to problem solving

- start with the problem to be solved (i.e., the top)
- break that problem down into smaller pieces and solve
- continue breaking down until reach base/trivial case (i.e., the bottom)

divide and conquer works well when the pieces can be solved independently

- e.g., merge sort – sorting each half can be done independently, no overlap

what about Fibonacci numbers? 1, 1, 2, 3, 5, 8, 13, 21, ...

```
public static int fib(int n) {
    if (n <= 1) {
        return 1;
    }
    else {
        return fib(n-1) + fib(n-2);
    }
}
```

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Top-down vs. bottom-up

divide and conquer is a horrible way of finding Fibonacci numbers

- the recursive calls are NOT independent; redundencies build up

```
public static int fib(int n) {
    if (n <= 1) {
        return 1;
    }
    else {
        return fib(n-1) + fib(n-2);
    }
}
```

```
fib(5)
  fib(4) + fib(3)
  fib(3) + fib(2) + fib(2) + fib(1)
  . . . . .
  . . . . .
  . . . . .
```

in this case, a bottom-up solution makes more sense

- start at the base cases (the bottom) and work up to the desired number
- requires remembering the previous two numbers in the sequence

```
public static int fib(int n) {
    int prev = 1, current = 1;
    for (int i = 1; i < n; i++) {
        int next = prev + current;
        prev = current;
        current = next;
    }
    return current;
}
```

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Dynamic programming

dynamic programming is a bottom-up approach to solving problems

- start with smaller problems and build up to the goal, storing intermediate solutions as needed
- applicable to same types of problems as divide & conquer, but bottom-up
- usually more effective than top-down if the parts are not completely independent (thus leading to redundancy)

example: binomial coefficient $C(n, k)$ is relevant to many problems

- the number of ways can you select k lottery balls out of n
- the number of birth orders possible in a family of n children where k are sons
- the number of acyclic paths connecting 2 corners of an $k \times (n-k)$ grid
- the coefficient of the $x^k y^{n-k}$ term in the polynomial expansion of $(x + y)^n$

$$C(n, k) \equiv \binom{n}{k} \equiv \frac{n!}{k!(n-k)!}$$

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Example: binary coefficient

while easy to define, a binomial coefficient is difficult to compute

e.g, 6 number lottery with 49 balls $\rightarrow 49!/6!43!$

$49! = 608,281,864,034,267,560,872,252,163,321,295,376,887,552,831,379,210,240,000,000,000$

could try to get fancy by canceling terms from numerator & denominator

- can still end up with individual terms that exceed integer limits

a computationally easier approach makes use of the following recursive relationship

$$\binom{n}{k} \equiv \binom{n-1}{k-1} + \binom{n-1}{k}$$

e.g., to select 6 lottery balls out of 49, partition into:

selections that include 1
(must select 5 out of remaining 48)

+

selections that don't include 1
(must select 6 out of remaining 48)

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Example: binomial coefficient

could use straight divide&conquer to compute based on this relation

```
/**
 * Calculates n choose k (using divide-and-conquer)
 * @param n the total number to choose from (n > 0)
 * @param k the number to choose (0 <= k <= n)
 * @return n choose k (the binomial coefficient)
 */
public static int binomial(int n, int k) {
    if (k == 0 || n == k) {
        return 1;
    }
    else {
        return binomial(n-1, k-1) + binomial(n-1, k);
    }
}
```

however, this will take a long time or exceed memory due to redundant work

$$\binom{47}{4} + \binom{47}{5} + \binom{49}{6} + \binom{47}{5} + \binom{47}{6}$$

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Dynamic programming solution

could instead work bottom-up, filling a table starting with the base cases (when $k = 0$ and $n = k$)

	0	1	2	3	...	k
0	1					
1	1	1				
2	1		1			
3	1			1		
...	
n	1					1

```

/**
 * Calculates n choose k (using dynamic programming)
 * @param n the total number to choose from (n > 0)
 * @param k the number to choose (0 <= k <= n)
 * @return n choose k (the binomial coefficient)
 */
public static int binomial(int n, int k) {
    if (n < 2) {
        return 1;
    }
    else {
        int bin[][] = new int[n+1][n+1]; // CONSTRUCT A TABLE TO STORE
        for (int r = 0; r <= n; r++) { // COMPUTED VALUES
            for (int c = 0; c <= r && c <= k; c++) {
                if (c == 0 || c == r) {
                    bin[r][c] = 1; // ENTER 1 IF BASE CASE
                }
                else {
                    bin[r][c] = bin[r-1][c-1] + bin[r-1][c]; // OTHERWISE, USE FORMULA
                }
            }
        }
        return bin[n][k]; // ANSWER IS AT bin[n][k]
    }
}
    
```

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Dynamic programming & caching

when calculating $C(n,k)$, the entire table must be filled in (up to the n th row and k th column)

→ working bottom-up from the base cases does not waste any work

	0	1	2	3	...	k
0	1					
1	1	1				
2	1		1			
3	1			1		
...	
n	1					1

for many problems, this is not the case

- solving a problem may require only a subset of smaller problems to be solved
- constructing a table and exhaustively working up from the base cases could do lots of wasted work
- can dynamic programming still be applied?

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Example: programming contest problem

November 13, 2004

ACM North Central North America Regional Programming Contest

Problem 1

Problem 1: Breaking a Dollar

Using only the U. S. coins worth 1, 5, 10, 25, 50, and 100 cents, there are exactly 293 ways in which one U. S. dollar can be represented. Canada has no coin with a value of 50 cents, so there are only 243 ways in which one Canadian dollar can be represented. Suppose you are given a new set of denominations for the coins (each of which we will assume represents some integral number of cents less than or equal to 100, but greater than 0). In how many ways could 100 cents be represented?

Input

The input will contain multiple cases. The input for each case will begin with an integer N (at least 1, but no more than 10) that indicates the number of unique coin denominations. By *unique* it is meant that there will not be two (or more) different coins with the same value. The value of N will be followed by N integers giving the denominations of the coins.

Input for the last case will be followed by a single integer -1.

Output

For each case, display the case number (they start with 1 and increase sequentially) and the number of different combinations of those coins that total 100 cents. Separate the output for consecutive cases with a blank line.

Sample Input

```
6 1 5 10 25 50 100
5 1 5 10 25 100
-1
```

Output for the Sample Input

```
Case 1: 293 combinations of coins
Case 2: 243 combinations of coins
```

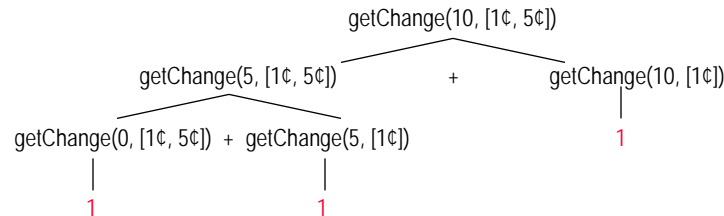
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Divide & conquer approach

let `getChange(amount, coinList)` represent the number of ways to get an amount using the specified list of coins

```
getChange(amount, coinList) =
  getChange(amount-biggestCoinValue, coinList) // # of ways that use at least
  +                                             // one of the biggest coin
  getChange(amount, coinList-biggestCoin) // # of ways that don't
                                             // involve the biggest coin
```

e.g., suppose want to get 10¢ using only pennies and nickels



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Caching

we could use dynamic programming and solve the problem bottom up

- however, consider `getChange(100, [1¢, 5¢, 10¢, 25¢])`
- would we ever need to know `getChange(99, [1¢, 5¢, 10¢, 25¢])` ?
`getChange(98, [1¢, 5¢, 10¢, 25¢])` ?
`getChange(73, [1¢, 5¢])` ?

when exhaustive bottom-up would yield too many wasted cases, dynamic programming can instead utilize top-down with *caching*

- create a table (as in the exhaustive bottom-up approach)
- however, fill the table in using a top-down approach
that is, execute a top-down divide and conquer solution, but store the solutions to subproblems in the table as they are computed
- before recursively solving a new subproblem, first check to see if its solution has already been cached
- avoids the duplication of pure top-down
- avoids the waste of exhaustive bottom-up (only solves relevant subproblems)

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ChangeMaker with caching

```
public class ChangeMaker {
    private List<Integer> coins;

    private static final int MAX_AMOUNT = 100;
    private static final int MAX_COINS = 10;
    private int[][] remember;

    public ChangeMaker(List<Integer> coins) {
        this.coins = coins;

        this.remember = new int[ChangeMaker.MAX_AMOUNT+1][ChangeMaker.MAX_COINS];
        for (int r = 0; r < ChangeMaker.MAX_AMOUNT+1; r++) {
            for (int c = 0; c < ChangeMaker.MAX_COINS; c++) {
                this.remember[r][c] = -1;
            }
        }
    }

    public int getChange(int amount) {
        return this.getChange(amount, this.coins.size()-1);
    }

    private int getChange(int amount, int maxCoinIndex) {
        if (maxCoinIndex < 0 || amount < 0) {
            return 0;
        }
        else if (this.remember[amount][maxCoinIndex] == -1) {
            if (amount == 0) {
                this.remember[amount][maxCoinIndex] = 1;
            }
            else {
                this.remember[amount][maxCoinIndex] =
                    this.getChange(amount-this.coins.get(maxCoinIndex), maxCoinIndex) +
                    this.getChange(amount, maxCoinIndex-1);
            }
        }
        return this.remember[amount][maxCoinIndex];
    }
}
```

as each subproblem is solved, its solution is stored in a table
each call to `getChange` checks the table first before recursing

with caching, even the worst case is fast:
6,292,069 combinations

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Algorithmic approaches summary

divide & conquer: tackles a complex problem by breaking it into smaller pieces, solving each piece, and combining them into an overall solution

- often involves recursion, if the pieces are similar in form to the original problem
- applicable for any application that can be divided into independent parts

dynamic: bottom-up implementation of divide & conquer – start with the base cases and build up to the desired solution, storing results to avoid redundant computation

- usually more effective than top-down recursion if the parts are not completely independent (thus leading to redundancy)
- note: can implement by adding caching to top-down recursion

greedy: involves making a sequence of choices/actions, each of which simply looks best at the moment

- applicable when a solution is a sequence of moves & perfect knowledge is available

backtracking: involves making a sequence of choices/actions (similar to greedy), but stores alternatives so that they can be attempted if the current choices lead to failure

- more costly in terms of time and memory than greedy, but general-purpose
- worst case: will have to try every alternative and exhaust the search space