CSC 321: Data Structures

Fall 2018

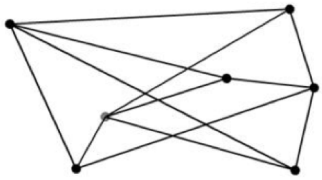
Graphs & search

- graphs, isomorphism
- simple vs. directed vs. weighted
- adjacency matrix vs. adjacency lists
- graph traversals
  depth-first search, breadth-first search
- finite automata

Graphs

trees are special instances of the more general data structure: graphs

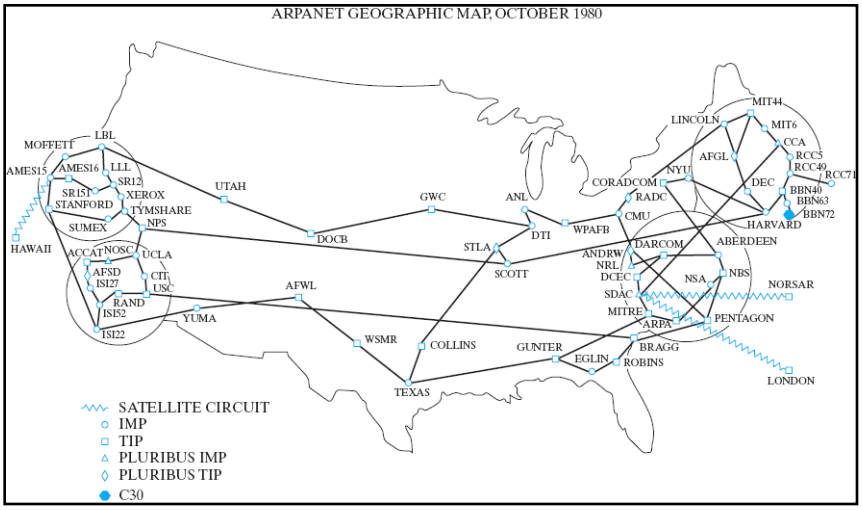
- informally, a graph is a collection of nodes/data elements with connections



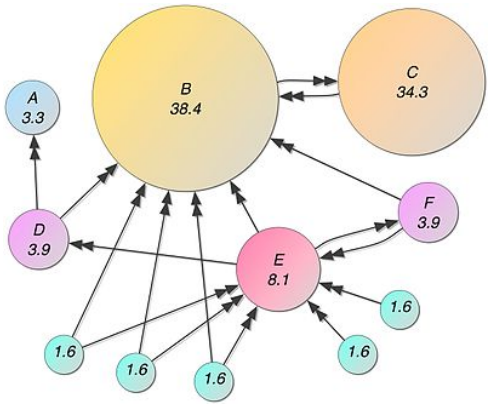
many real-world and CS-related applications rely on graphs

- the connectivity of computers in a network
- cities on a map, connected by roads
- cities on a airline schedule, connected by flights
- Facebook friends
- finite automata

Interesting graphs



Interesting graphs



Google pagerank algorithm

- the pagerank of a page is based on the number of pages that link to it and their pageranks

https://en.wikipedia.org/wiki/PageRank

Interesting graphs

TouchGraph Facebook Browser

- allows you to view all of your friends and groups as a graph

http://www.touchgraph.com/facebook

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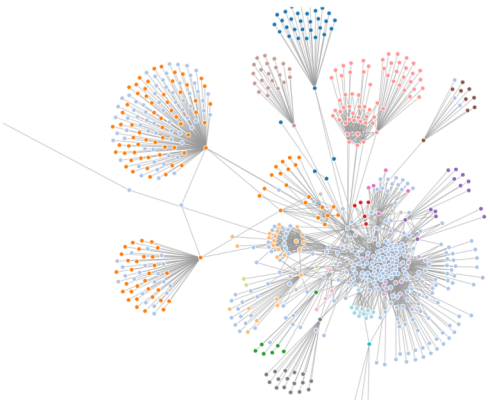
Interesting graphs



Enron email graph

- was used during prosecution to identify conspirators

http://www.contextsmith.com/blog/enron-emails-help-improve-project-communication/

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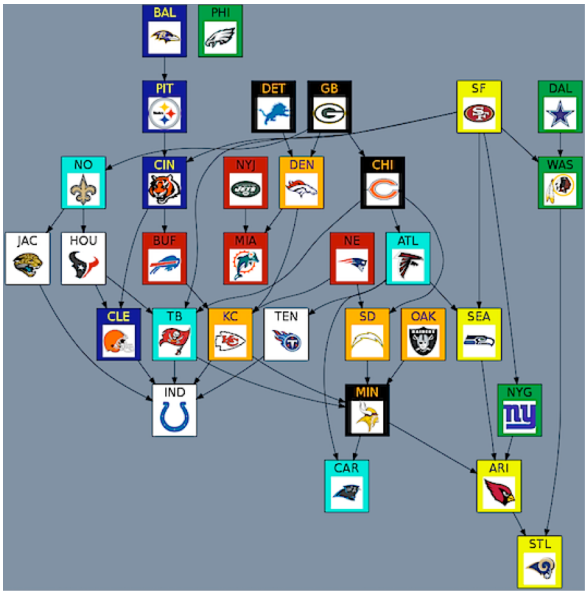
Interesting graphs



NFL 2011 @ week 11

- edges from one team to another show a clear beat path
- e.g., Chicago > Tampa > Indy

http://beatpaths.com/

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Formal definition

a *simple graph* G consists of a nonempty set V of vertices, and a set E of edges (unordered vertex pairs)

- can write simply as G = (V, E)

e.g., consider G = (V, E) where V = {a, b, c, d} and E = {{a,b}, {a, c}, {b,c}, {c,d}}
➔ DRAW IT!

graph terminology for G = (V, E)

- vertices u and v are *adjacent* if {u, v} ∈ E (i.e., connected by an edge)
- edge {u,v} is *incident* to vertices u and v
- the *degree* of v is the # of edges incident with it (or # of vertices adjacent to it)
- |V| = # of vertices |E| = # of edges
- a path between v_1 and v_n is a sequence of edges from E:
  [$\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{n-2}, v_{n-1}\}, \{v_{n-1}, v_n\}$]

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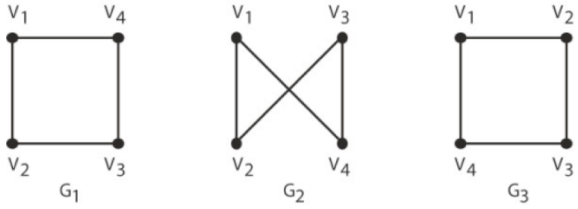
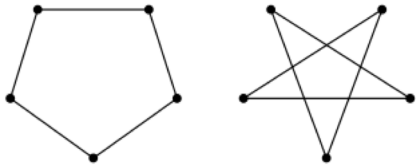
Depicting graphs

you can draw the same set of vertices & edges many different ways



are these different graphs?



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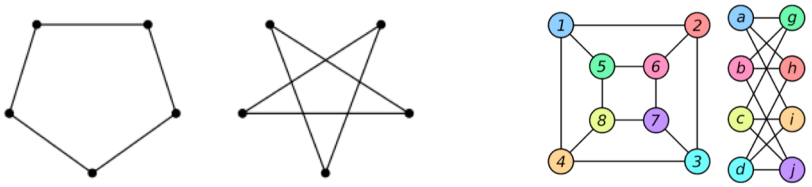
Isomorphism

how can we tell if two graphs are the same?

- $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are *isomorphic* if there is mapping $f:V_1 \rightarrow V_2$ such that $\{u,v\} \in E_1$ iff $\{f(u), f(v)\} \in E_2$
- i.e., G_1 and G2 are *isomorphic* if they are identical modulo a renaming of vertices



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Simple vs. directed?

in a simple graph, edges are presumed to be bidirectional

- {u,v} ∈ E is equivalent to saying {v,u} ∈ E

for many real-world applications, this is appropriate

- if computer c_1 is connected to computer c_2, then c_2 is connected to c_1
- if person p_1 is Facebook friends with person p_2, then p_2 is friends with p_1

however, in other applications the connection may be 1-way

- a flight from city c_1 to city c_2 does not imply a return flight from c_2 to c_1
- roads connecting buildings in a town can be 1-way (with no direct return route)
- NFL beat paths are clearly directional

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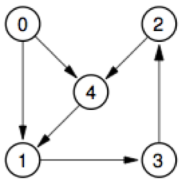
Directed graphs

a *directed graph* or *digraph* G consists of a nonempty set V of vertices, and a set E of edges (ordered vertex pairs)

- draw edge (u,v) as an arrow pointing from vertex u to vertex v



e.g., consider G = (V, E) where V = {a, b, c, d} and E = {(a,b), (a, c), (b,c), (c,d)}

➔DRAW IT!

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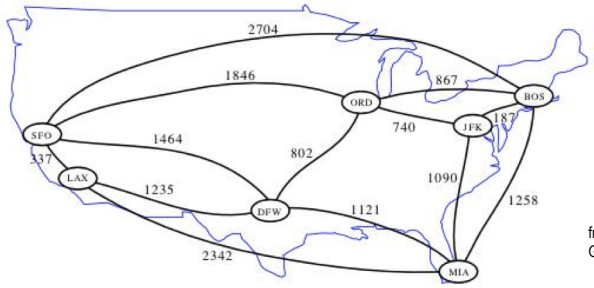
Weighted graphs

a *weighted graph* G = (V, E) is a graph/digraph with an additional weight function $\omega: E \rightarrow \Re$

- draw edge (u,v) labeled with its weight



from *Algorithm Design* by Goodrich & Tamassia

- shortest path from BOS to SFO?
- does the fewest legs automatically imply the shortest path in terms of miles?

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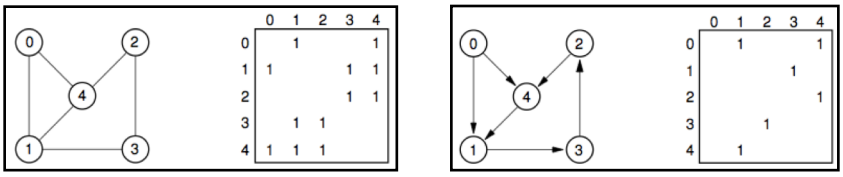
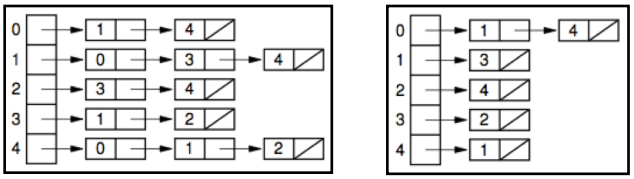
Graph data structures

- an *adjacency matrix* is a 2-D array, indexed by the vertices
  for graph G, adjMatrix[u][v] = 1 iff {u,v} ∈ E (or weights in place of 1s)



- an *adjacency list* is an array of linked lists
  for graph G, adjList[u] contains v iff {u,v} ∈ E (or weights in addition to vertex)



note: our HashMap of HashMaps was a smart implementation of an adjacency list

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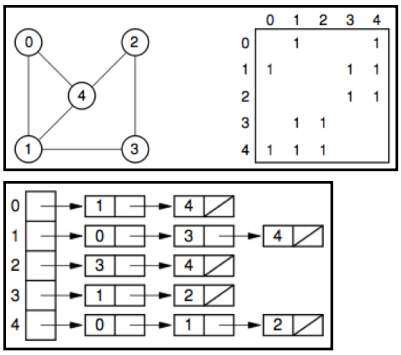
Matrix vs. list



space tradeoff

- adjacency matrix requires $O(|V|^2)$ space
- adjacency list requires $O(|V| + |E|)$ space

which is better?

- it depends
- if the graph is sparse (few edges), then $|V|+|E| < |V|^2$ → adjacency list
- if the graph is dense, i.e., $|E|$ is $O(|V|^2)$, then $|V|^2 < |V|+|E|$ → adjacency matrix

many graph algorithms involve visiting every adjacent neighbor of a node

- adjacency list makes this efficient, so tends to be used more in practice

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Java implementations

to allow for alternative implementations, we will define a Graph interface

```
import java.util.Set;

public interface Graph<E> {
    public boolean isAdjacentTo(E v1, E v2);
    public Set<E> getAllAdjacent(E v);
}
```

many other methods are possible, but these suffice for now

from this, we can implement simple graphs or directed graphs

- can utilize an adjacency matrix or an adjacency list

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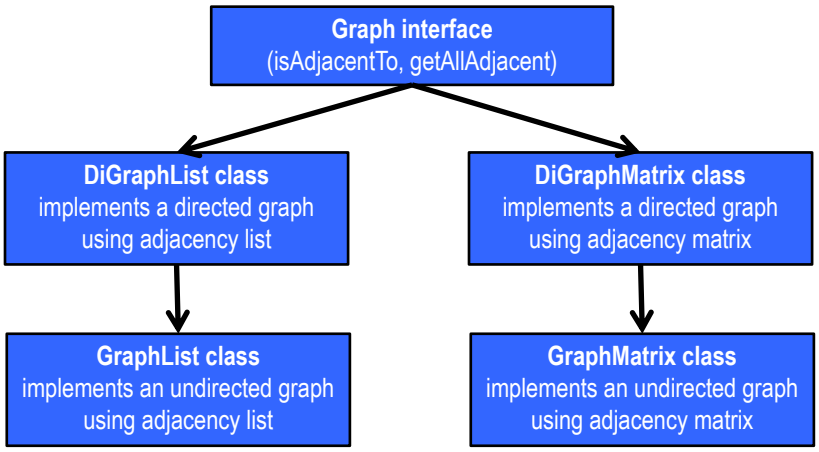
Class hierarchy

in addition to the Graph interface, can utilize inheritance to simplify the development of the graph variants



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DiGraphList

```
public class DiGraphList<E> implements Graph<E>{
  private Map<E, Set<E>> adjLists;

  public DiGraphList(List<Pair<E, E>> edges) {
    this.adjLists = new HashMap<E, Set<E>>();
    for (Pair<E, E> edgePair : edges) {
      this.addEdge(edgePair.getKey(), edgePair.getValue());
    }
  }

  protected void addEdge(E v1, E v2) {
    if (!this.adjLists.containsKey(v1)) {
      this.adjLists.put(v1, new HashSet<E>());
    }
    this.adjLists.get(v1).add(v2);
  }

  public boolean isAdjacentTo(E v1, E v2) {
    return this.adjLists.containsKey(v1) &&
           this.getAllAdjacent(v1).contains(v2);
  }

  public Set<E> getAllAdjacent(E v) {
    if (!this.adjLists.containsKey(v)) {
      return new HashSet<E>();
    }
    return this.adjLists.get(v);
  }
}
```

we could implement the adjacency list using a standard array of LinkedLists

- conceptually equivalent but simpler approach: use a HashMap to map each vertex to a Set of adjacent vertices
- (but it does waste memory)
- constructor takes a list of edges (stored as Pairs)
- helper method `addEdge` handles storing an edge in the map

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DiGraphMatrix

```
public class DiGraphMatrix<E> implements Graph<E>{
  private E[] vertices;
  private Map<E, Integer> lookup;
  private boolean[][] adjMatrix;

  public DiGraphMatrix(List<Pair<E, E>> edges) {
    Set<E> vertexSet = new HashSet<E>();
    for (Pair<E, E> edgePair : edges) {
      vertexSet.add(edgePair.getKey());
      vertexSet.add(edgePair.getValue());
    }

    this.vertices = (E[])(new Object[vertexSet.size()]);
    this.lookup = new HashMap<E, Integer>();

    int index = 0;
    for (E nextVertex : vertexSet) {
      this.vertices[index] = nextVertex;
      lookup.put(nextVertex, index);
      index++;
    }

    this.adjMatrix = new boolean[index][index];

    for (Pair<E, E> edgePair : edges) {
      this.addEdge(edgePair.getKey(), edgePair.getValue());
    }
  }

  ...
```

adjacency matrix implementation is tricky in Java

- have to first determine the set of vertices in order to create the table
- since Java arrays are only accessible via indices, have to map to/from labels

- `vertices` array converts index → label
- `lookup` map converts label → index

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DiGraphMatrix

```
  ...

  protected void addEdge(E v1, E v2) {
    Integer index1 = this.lookup.get(v1);
    Integer index2 = this.lookup.get(v2);
    if (index1 != null && index2 != null) {
      this.adjMatrix[index1][index2] = true;
    }
  }

  public boolean isAdjacentTo(E v1, E v2) {
    Integer index1 = this.lookup.get(v1);
    Integer index2 = this.lookup.get(v2);
    return index1 != null && index2 != null &&
           this.adjMatrix[index1][index2];
  }

  public Set<E> getAllAdjacent(E v) {
    Integer row = this.lookup.get(v);

    Set<E> neighbors = new HashSet<E>();
    if (row != null) {
      for (int c = 0; c < this.adjMatrix.length; c++) {
        if (this.adjMatrix[row][c]) {
          neighbors.add(this.vertices[c]);
        }
      }
    }
    return neighbors;
  }
}
```

`addEdge` is easy

- lookup indices of vertices
- then store `true` in matrix at row/column

`isAdjacentTo` also easy

- lookup indices of vertices
- then access row/column

`getAllAdjacent` is more work

- lookup index of vertex
- traverse row
- collect all adjacent vertices in a set

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GraphMatrix & GraphList

can utilize inheritance to implement simple graph variants

- can utilize the same internal structures, just add edges in both directions

```
public class GraphList<E> extends DiGraphList<E> {
  public GraphList(List<Pair<E, E>> edges) {
    super(edges);
  }

  protected void addEdge(E v1, E v2) {
    super.addEdge(v1, v2);
    super.addEdge(v2, v1);
  }
}
```

```
public class GraphMatrix<E> extends DiGraphMatrix<E> {
  public GraphMatrix(List<Pair<E, E>> edges) {
    super(edges);
  }

  protected void addEdge(E v1, E v2) {
    super.addEdge(v1, v2);
    super.addEdge(v2, v1);
  }
}
```

constructors simply call the superclass constructors

override the `addEdge` methods so that they add edges in both directions (using the superclass `addEdge`)

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Graph traversal

many applications involve systematically searching through a graph

- starting at some vertex, want to traverse the graph and inspect/process vertices along paths

e.g., list all the computers on the network

e.g., find a sequence of flights that get the customer from BOS to SFO (perhaps with some property?)

similar to tree traversals, we can follow various patterns

- recall for trees: inorder, preorder, postorder

for arbitrary graphs, two main alternatives

- depth-first search (DFS): boldly follow a single path as long as possible
- breadth-first search (BFS): tentatively try all paths level-by-level

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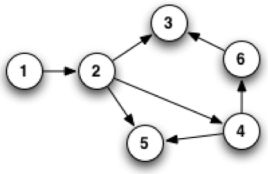
DFS vs. BFS



consider the following digraph

- suppose we wanted to find a path from 1 to 5

depth-first search would:

▪ start with the starting vertex	1
▪ select an adjacent vertex from there	1 → 2
▪ select an adjacent vertex from there	1 → 2 → 3
▪ dead-end, so back up	1 → 2
▪ select a different adjacent vertex	1 → 2 → 4
▪ select an adjacent vertex from there	1 → 2 → 4 → 5

breadth-first search would:

▪ start with the starting vertex	1
▪ expand to length-1 paths	1 → 2
▪ expand to length-2 paths	1 → 2 → 3
	1 → 2 → 4
	1 → 2 → 5

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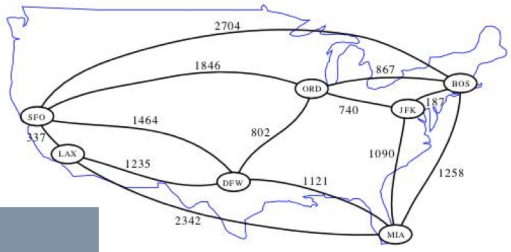
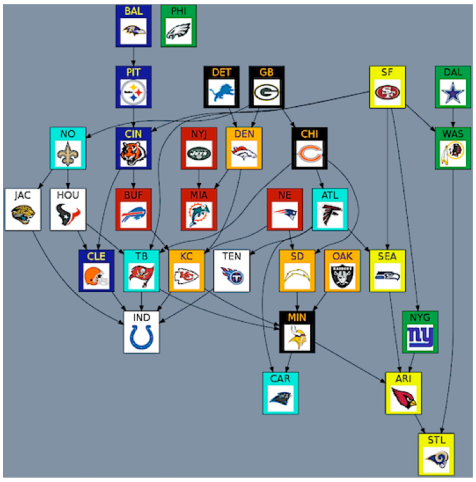
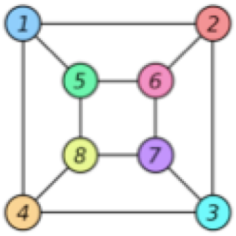
Bigger examples

DFS? BFS?







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DFS implementation

DFS can be implemented recursive backtracking (more in CSC421)

```
public static <E> List<E> DFSrec(Graph<E> g, E start, E goal) {
    List<E> path = new ArrayList<E>();
    path.add(start);
    if (GraphSearch.DFSrec(g, path, goal)) {
        return path;
    }
    return null;
}

private static <E> boolean DFSrec(Graph<E> g,
                                  List<E> pathSoFar, E goal) {
    E lastVertex = pathSoFar.get(pathSoFar.size()-1);
    if (lastVertex.equals(goal)) {
        return true;
    }
    else {
        for (E adj : g.getAllAdjacent(lastVertex)) {
            if (!pathSoFar.contains(adj)) {
                pathSoFar.add(adj);
                if (GraphSearch.DFSrec(g, pathSoFar, goal)) {
                    return true;
                }
                pathSoFar.remove(pathSoFar.size()-1);
            }
        }
        return false;
    }
}
```

recursive method is passed path so far and goal state

- BASE CASE: when path so far ends in goal state
- RECURSIVE: extend the path for each adjacent vertex and recurse
- to avoid cycles, don't extend if the adjacent vertex is already in the path

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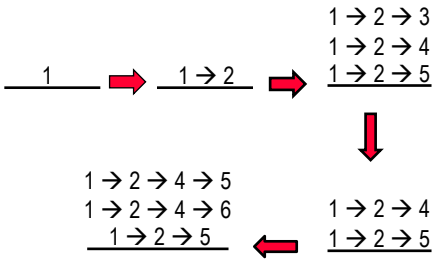
Stacks & Queues

DFS can alternatively be implemented using a stack of paths

- initially contains a single path (w/ start)
- at each step, pop the path at the top of the stack
- extend that path for each adjacent vertex, and push that extended path on the stack



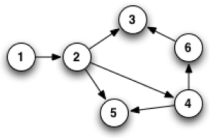
BFS is most naturally implemented using a queue of paths

- initially contains a single path (w/ start)
- at each step, remove the path at the front of the queue
- extend that path for each adjacent vertex, and add that extended path to the back of the queue

[1]

[1 → 2]

[1→ 2 → 5, 1 → 2 → 4, 1 → 2 → 3]



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Implementations

```
public static <E> List<E> DFS(Graph<E> g, E start, E goal) {
    List<E> startPath = new ArrayList<E>();
    startPath.add(start);

    Stack<List<E>> paths = new Stack<List<E>>();
    paths.push(startPath);

    while (!paths.isEmpty()) {
        List<E> pathToExtend = paths.pop();
        E current = pathToExtend.get(pathToExtend.size()-1);
        if (current.equals(goal)) {
            return pathToExtend;
        }
        else {
            for (E adj : g.getAllAdjacent(current)) {
                if (!pathToExtend.contains(adj)) {
                    List<E> copy = new ArrayList<E>(pathToExtend);
                    copy.add(adj);
                    paths.push(copy);
                }
            }
        }
    }
    return null;
}
```

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Implementations

```
public static <E> List<E> BFS(Graph<E> g, E start, E goal) {
    List<E> startPath = new ArrayList<E>();
    startPath.add(start);

    Queue<List<E>> paths = new LinkedList<List<E>>();
    paths.add(startPath);

    while (!paths.isEmpty()) {
        List<E> pathToExtend = paths.remove();
        E current = pathToExtend.get(pathToExtend.size()-1);
        if (current.equals(goal)) {
            return pathToExtend;
        }
        else {
            for (E adj : g.getAllAdjacent(current)) {
                if (!pathToExtend.contains(adj)) {
                    List<E> copy = new ArrayList<E>(pathToExtend);
                    copy.add(adj);
                    paths.add(copy);
                }
            }
        }
    }
    return null;
}
```

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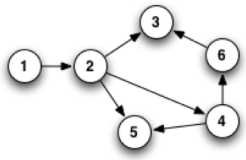
DFSrec vs. DFS vs. BFS

```
public static void main(String[] args) throws java.io.FileNotFoundException {
    List<Pair<Integer, Integer>> edges = new ArrayList<Pair<Integer, Integer>>();
    edges.add(new Pair<Integer, Integer>(1, 2));
    edges.add(new Pair<Integer, Integer>(2, 3));
    edges.add(new Pair<Integer, Integer>(2, 4));
    edges.add(new Pair<Integer, Integer>(2, 5));
    edges.add(new Pair<Integer, Integer>(4, 5));
    edges.add(new Pair<Integer, Integer>(4, 6));
    edges.add(new Pair<Integer, Integer>(6, 3));

    Graph<Integer> g = new DiGraphList<Integer>(edges);

    System.out.println("DFSrec " + GraphSearch.DFSrec(g, 1, 3));
    System.out.println("DFSrec " + GraphSearch.DFSrec(g, 1, 5));
    System.out.println();
    System.out.println("DFS " + GraphSearch.DFS(g, 1, 3));
    System.out.println("DFS " + GraphSearch.DFS(g, 1, 5));
    System.out.println();
    System.out.println("BFS " + GraphSearch.BFS(g, 1, 3));
    System.out.println("BFS " + GraphSearch.BFS(g, 1, 5));
}
```




```
DFSrec [1, 2, 3]
DFSrec [1, 2, 4, 5]

DFS [1, 2, 4, 6, 3]
DFS [1, 2, 5]

BFS [1, 2, 3]
BFS [1, 2, 5]
```

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HW6: PathFinder

```
public List<StateLabel> findPath(StateLabel startState, StateLabel endState) {
    List<StateLabel> startPath = new ArrayList<StateLabel>();
    startPath.add(startState);

    Queue<List<StateLabel>> paths = new LinkedList<List<StateLabel>>();
    paths.add(startPath);

    while (!paths.isEmpty()) {
        List<StateLabel> shortestPath = paths.remove();
        StateLabel current = shortestPath.get(shortestPath.size()-1);
        if (current.equals(endState)) {
            return shortestPath;
        }
        else {
            for (StateLabel s : this.getAllAdjacentStates(current)) {
                if (!shortestPath.contains(s)) {
                    List<StateLabel> copy = new ArrayList<StateLabel>(shortestPath);
                    copy.add(s);
                    paths.add(copy);
                }
            }
        }
    }
    return null;
}
```

HW6 utilizes a modified BFS to find shortest paths in a FSM

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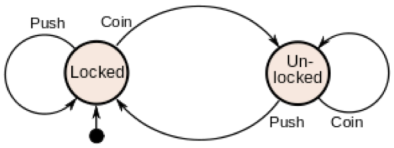
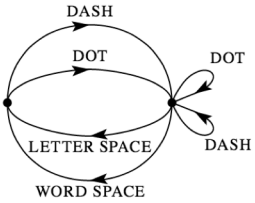
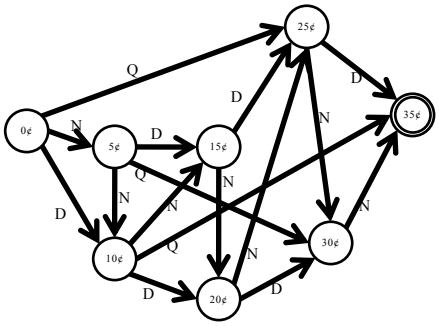
Finite State Machines (revisited)

recall: a *Finite State Machine* (a.k.a. *Finite Automaton*) defines a finite set of states (i.e., vertices) along with transitions between those states (i.e., edges)







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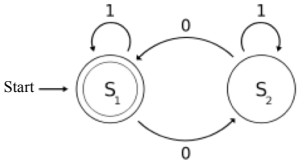
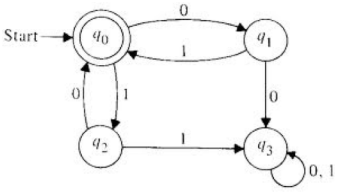
Recognition/acceptance

we say that a FSM *recognizes a pattern* if all sequences that meet that pattern terminate in an accepting state (drawn with double circle)





accepts
- 1
- 00
- 0101
- 0001110

in general?

accepts
- 10
- 01
- 1010
- 100110

in general?

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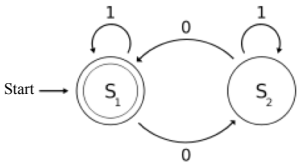
Regular expressions

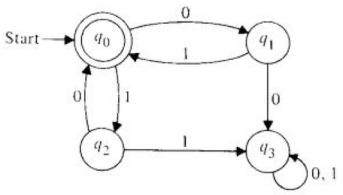
alternatively, we say that a FSM *defines a language* made up of the sequences accepted by the FSM

- a language is known as a *regular language* if there exists a FSM that accepts it
- a *regular expression* denotes a regular language using * for arbitrary repetition and + for OR

L_1 = 1*(01*0)*1*



L_2 = (01 + 10)*



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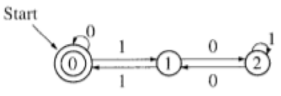
Old GRE CS exam – sample question

28.



State 0 is both the starting state and the accepting state.

Each of the following is a regular expression that denotes a subset of the language recognized by the automaton above EXCEPT

(A) 0*(11)*0*
(B) 0*1(10*1)*1
(C) 0*1(10*1)*10*
(D) 0*1(10*1)0(100)*
(E) (0*1(10*1)*10* + 0*)*

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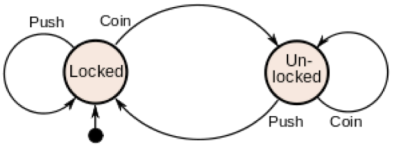
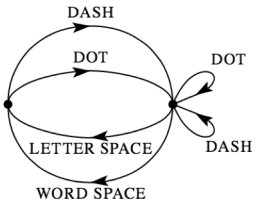
Regular language examples



if `locked` is initial state and `unlocked` is goal state:

```
Coin (Push Push* Coin)* Coin*
```



if `betweenLetters` is initial and goal state:

```
((Dot + Dash) (Dot + Dash)* (Lspace + Wspace))*
```

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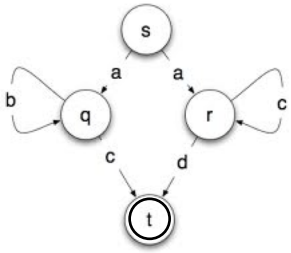
Deterministic vs. nondeterministic

the examples so far are *deterministic*, i.e., each transition is unambiguous

- a FSM can also be *nondeterministic*, if there are multiple transitions from a state with different labels/weights
- a nondeterministic FSM accepts a sequence if there exists a path that accepts
- nondeterministic FSMs are especially useful for defining complex regular languages



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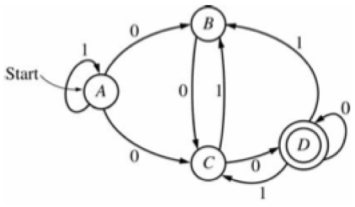
MFT CS exam – sample question



1. If D is the accepting state of the nondeterministic finite automaton above, which of the following does the automaton accept?

   (A) 001
   (B) 1101
   (C) 01100
   (D) 000110
   (E) 100100

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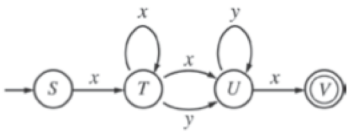
GRE CS exam – sample question

Consider the following nondeterministic finite state automaton over alphabet $\{x, y\}$ with start state S.



4. Which of the following is the regular expression corresponding to the automaton above?

   (A) $xxx + yyx$

   (B) x^3y^2x

   (C) x^*y^*x

   (D) $xx^*(x + y)y^*x$

   (E) x^*xy^*x

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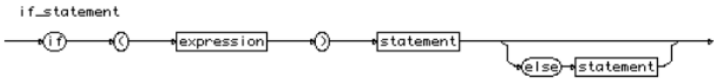
Uses of FSM

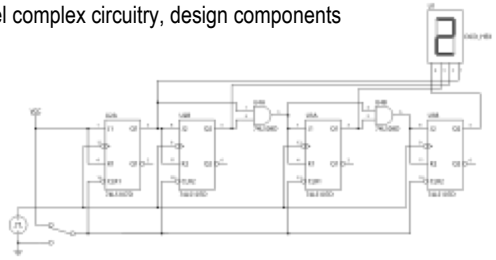
FSM machines are useful in many areas of CS

- designing software for controlling devices (e.g., vending machine, elevator, …)
  first identify states, constructing a table of state transitions, then implement in code

- parsing symbols for programming languages or text processing
  may define the language using rules or via a regular expression, then build a FSM to parse the characters into tokens & symbols



- designing circuitry
  commonly used to model complex circuitry, design components



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