CSC 321: Data Structures

Fall 2018

Algorithm analysis, searching and sorting

- best vs. average vs. worst case analysis
- big-Oh analysis (intuitively)
- analyzing searches & sorts
- general rules for analyzing algorithms
- analyzing recursion recurrence relations
- specialized sorts
- big-Oh analysis (formally), big-Omega, big-Theta

Algorithm efficiency

when we want to classify the efficiency of an algorithm, we must first identify the costs to be measured

▪ memory used?	sometimes relevant, but not usually driving force
▪ execution time?	dependent on various factors, including computer specs
▪ # of steps	somewhat generic definition, but most useful

to classify an algorithm's efficiency, first identify the steps that are to be measured

e.g., for searching: # of inspections, …
for sorting: # of inspections, # of swaps, # of inspections + swaps, …

must focus on key steps (that capture the behavior of the algorithm)

- e.g., for searching: there is overhead, but the work done by the algorithm is dominated by the number of inspections

Best vs. average vs. worst case

when measuring efficiency, you need to decide what case you care about

- best case: usually not of much practical use
  the best case scenario may be rare, certainly not guaranteed

- average case: can be useful to know
  on average, how would you expect the algorithm to perform
  can be difficult to analyze – must consider all possible inputs and calculate the average performance across all inputs

- worst case: most commonly used measure of performance
  provides upper-bound on performance, guaranteed to do no worse

sequential search:	best?	average?	worst?
binary search:	best?	average?	worst?

note: best ≠ small, worst ≠ big   best/worst cases are relative to arbitrary size N

3

Big-Oh (intuitively)

intuitively: an algorithm is O(f(N)) if the # of *steps* involved in solving a problem of size N has f(N) as the dominant term

O(N):	5N	3N + 2	N/2 – 20
$O(N^2)$:	N^2	$N^2 + 100$	$10N^2 – 5N + 100$

…

why aren't the smaller terms important?

- big-Oh is a "long-term" measure
- when N is sufficiently large, the largest term dominates

consider $f_1(N) = 300*N$ (a very steep line) & $f_2(N) = ½*N^2$ (a very gradual quadratic)

in the short run (i.e., for small values of N), $f_1(N) > f_2(N)$
  e.g., $f_1(10) = 300*10 = 3,000 > 50 = ½*10^2 = f_2(10)$
in the long run (i.e., for large values of N), $f_1(N) < f_2(N)$
  e.g., $f_1(1,000) = 300*1,000 = 300,000 < 500,000 = ½*1,000^2 = f_2(1,000)$

4

Big-Oh and rate-of-growth

big-Oh classifications capture rate of growth

- for an O(N) algorithm, doubling the problem size doubles the amount of work

  e.g., suppose Cost(N) = 5N – 3
  - Cost(s) = 5s – 3
  - Cost(2s) = 5(2s) – 3 = 10s - 3

- for an O(N log N) algorithm, doubling the problem size more than doubles the amount of work

  e.g., suppose Cost(N) = 5N log N + N
  - Cost(s) = 5s log s + s
  - Cost(2s) = 5(2s) log (2s) + 2s = 10s(log(s)+1) + 2s = 10s log s + 12s

- for an O(N^2) algorithm, doubling the problem size quadruples the amount of work

  e.g., suppose Cost(N) = $5N^2 – 3N + 10$
  - Cost(s) = $5s^2 – 3s + 10$
  - Cost(2s) = $5(2s)^2 – 3(2s) + 10 = 5(4s^2) – 6s + 10 = 20s^2 – 6s + 10$

5

Big-Oh of searching/sorting

sequential search: worst case cost of finding an item in a list of size N

- may have to inspect every item in the list

Cost(N) = N inspections + overhead
→ O(N)

selection sort: cost of sorting a list of N items

- make N-1 passes through the list, comparing all elements and performing one swap

Cost(N) = (1 + 2 + 3 + … + N-1) comparisons + N-1 swaps + overhead
= N*(N-1)/2 comparisons + N-1 swaps + overhead
= ½ N^2 – ½ N comparisons + N-1 swaps + overhead
→ O(N^2)

6

General rules for analyzing algorithms

1. for loops: the running time of a for loop is at most

   running time of statements in loop × number of loop iterations

```
for (int i = 0; i < N; i++) {
    sum += nums[i];
}
```

2. nested loops: the running time of a statement in nested loops is

   running time of statement in loop × product of sizes of the loops

```
for (int i = 0; i < N; i++) {
    for (int j = 0; j < M; j++) {
        nums1[i] += nums2[j] + i;
    }
}
```

7

General rules for analyzing algorithms

3. consecutive statements: the running time of consecutive statements is

   sum of their individual running times

```
int sum = 0;
for (int i = 0; i < N; i++) {
    sum += nums[i];
}
double avg = (double)sum/N;
```

4. if-else: the running time of an if-else statement is at most

   running time of the test + maximum running time of the if and else cases

```
if (isSorted(nums)) {
    index = binarySearch(nums, desired);
}
else {
    index = sequentialSearch(nums, desired);
}
```

8

EXAMPLE: finding all anagrams of a word (approach 1)

```
for each possible permutation of the word
    • generate the next permutation
    • test to see if contained in the dictionary
    • if so, add to the list of anagrams
```

efficiency of this approach, where L is word length & D is dictionary size?

```
for each possible permutation of the word
    • generate the next permutation
```

→ O(L), *assuming a smart encoding*

```
    • test to see if contained in the dictionary
```

→ O(D), *assuming sequential search*

```
    • if so, add to the list of anagrams
```

→O(1)

since L! different permutations, will loop L! times

➔ $O(L! \times (L + D + 1))$ ➔ $O(L! \times D)$

note: 6! = 720, 7! = 5,040, 8! = 40,320, 9! = 362,880, 10! = 3,628,800, 11! = 39,916,800

9

EXAMPLE: finding all anagrams of a word (approach 2)

```
sort letters of given word
traverse the entire dictionary, word by word
    • sort the next dictionary word
    • test to see if identical to sorted given word
    • if so, add to the list of anagrams
```

efficiency of this approach, where L is word length & D is dictionary size?

```
sort letters of given word
```

→ O(L log L), *assuming an efficient sort*

```
traverse the entire dictionary, word by word
    • sort the next dictionary word
```

→O(L log L), *assuming an efficient sort*

```
    • test to see if identical to sorted given word
```

→ O(L)

```
    • if so, add to the list of anagrams
```

→ O(1)

since dictionary is size D, will loop D times

➔ $O(L \log L + (D \times (L \log L + L + 1)))$ ➔ $O(L \log L \times D)$

10

Approach 1 vs. approach 2

clearly, approach 2 will be faster $O(L \log L \times D)$ vs. $O(L! \times D)$

- for a 5-letter word:

$5 \log 5 \times 117{,}000 \approx 12 \times 117{,}000 =$ 1,404,000

$5! \times 117{,}000 = 120 \times 117{,}000 =$ 14,040,000

- for a 10-letter word:

$10 \log 10 \times 117{,}000 \approx 33 \times 117{,}000 =$ 3,861,000

$10! \times 117{,}000 = 3{,}628{,}800 \times 117{,}000 =$ 424,569,600,000

approach 3: instead of sorting the letters in a word, count the number of a's, b's, c's, … and compare with counts from the other word EFFICIENCY?

11

Analyzing recursive algorithms

recursive algorithms can be analyzed by defining a *recurrence relation*:

cost of searching N items using binary search =
cost of comparing middle element + cost of searching correct half (N/2 items)

more succinctly: Cost(N) = Cost(N/2) + C

Cost(N) = Cost(N/2) + C	can unwind Cost(N/2)
= (Cost(N/4) + C) + C	
= Cost(N/4) + 2C	can unwind Cost(N/4)
= (Cost(N/8) + C) + 2C	
= Cost(N/8) + 3C	can continue unwinding
= …	(a total of $\log_2 N$ times)
= Cost(1) + $(\log_2 N)$*C	
= $C \log_2 N + C'$	where C' = Cost(1)

➔ O(log N)

12

Analyzing merge sort

cost of sorting N items using merge sort =
cost of sorting left half (N/2 items) + cost of sorting right half (N/2 items) +
cost of merging (N items)

more succinctly: $Cost(N) = 2Cost(N/2) + C_1N + C_2$

$Cost(N) = 2Cost(N/2) + C_1{*}N + C_2$	can unwind Cost(N/2)
$= 2(\ 2Cost(N/4) + C_1N/2 + C_2) + C_1N + C_2$	
$= 4Cost(N/4) + 2C_1N + 3C_2$	can unwind Cost(N/4)
$= 4(\ 2Cost(N/8) + C_1N/4 + C_2) + 2C_1N + 3C_2$	
$= 8Cost(N/8) + 3C_1N + 7C_2$	can continue unwinding
$= \ldots$	(a total of $\log_2 N$ times)
$= NCost(1) + (\log_2 N)C_1N + (N-1)\ C_2$	
$= C_1N \log_2 N + (C'+C_2)N - C_2$	where C' = Cost(1)

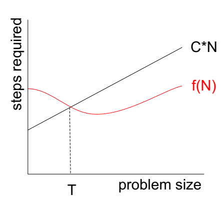
➔ O(N log N)

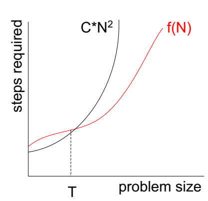
13

Big-Oh (slightly more formally)

more formally: an algorithm is O(f(N)) if, *after some point*, the # of steps can be bounded from above by a scaled f(N) function

O(N): if number of steps can eventually be bounded by a line
$O(N^2)$: if number of steps can eventually be bounded by a quadratic
…





"after some point" captures the fact that we only care about the long run

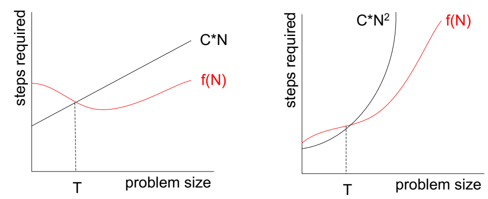
- for small values of N, the constants can make an O(N) algorithm do more work than an $O(N^2)$ algorithm
- but beyond some threshold size, the $O(N^2)$ will always do more work

e.g., $f_1(N) = 300N$ & $f_2(N) = ½\ N^2$ what threshold forces $f_1(N) \leq f_2(N)$?

14

Big-Oh (formally)

an algorithm is O(f(N)) if there exists a positive constant C & non-negative integer T such that for all $N \geq T$, # of steps required $\leq C*f(N)$



for example, selection sort:

N(N-1)/2 inspections + N-1 swaps = $(N^2/2 + N/2 - 1)$ steps

if we consider C = 1 and T = 1, then

$N^2/2 + N/2 - 1$	$\leq N^2/2 + N/2$	*since added 1 to rhs*
	$\leq N^2/2 + N(N/2)$	*since $1 \leq N$ at T and beyond*
	$= N^2/2 + N^2/2$	
	$= 1N^2$	➔ $O(N^2)$

in general, can use C = sum of positive terms, T = 1 (but other constants work too)

Exercises

consider an algorithm whose cost function is

$Cost(N) = 3N^2 - 12N + 5$

intuitively, we know this is $O(N^2)$

formally, what are values of C and T that meet the definition?

- an algorithm is $O(N^2)$ if there exists a positive constant C & non-negative integer T such that for all $N \geq T$, # of steps required $\leq C*N^2$

consider an algorithm whose cost function is

$Cost(N) = 12N^3 - 5N^2 + N - 300$

intuitively, we know this is $O(N^3)$

formally, what are values of C and T that meet the definition?

- an algorithm is $O(N^3)$ if there exists a positive constant C & non-negative integer T such that for all $N \geq T$, # of steps required $\leq C*N^3$

Exercise

consider a merge-3 sort algorithm

1. if the list contains 0 or 1 items, then done
2. otherwise, divide the list into thirds and recursively sort each third
3. then, merge the sorted thirds into a single sorted list

what is the recurrence relation for this algorithm?

closed (polynomial) form?

Big-Oh?

17

Specialized sorts

for general-purpose, comparable data, O(N log N) is optimal

- i.e., it is proven that there is no sorting algorithm better than O(N log N) for sorting arbitrary lists of elements (using only data comparisons)
- proof later

interestingly, you can do better *in special cases*

- if the range of potential data values is limited → frequency list
- if the data values can be compared lexicographically → radix sort

18

Frequency lists

suppose there is a fixed, reasonably-sized range of values

- such as years in the range 1900-2006

1975	2002	2006	2002	2005	1999	1950	1903	2006	2001	2006	1975	2003	1900	1980	1900

- construct a frequency array with |range| counters, one for each year

2	0	0	1	. . .	1	2	1	0	1	3
1900	1901	1902	1903	. . .	2001	2002	2003	2004	2005	2006

- then traverse and copy the appropriate values back to the list

1900	1900	1903	1950	1975	1975	1980	1999	2001	2002	2002	2003	2005	2006	2006	2006

big-Oh analysis?

Radix sort

suppose the values can be compared lexicographically (either character-by-character or digit-by-digit)

radix sort:

1. take the least significant char/digit of each value
2. sort the list based on that char/digit, but keep the order of values with the same char/digit
3. repeat the sort with each more significant char/digit

"ace"	"baa"	"cad"	"bee"	"bad"	"ebb"

most often implemented using a "bucket list"

- here, need one bucket for each possible letter
- copy all of the words ending in "a" in the 1st bucket, "b" in the 2nd bucket, …

"baa"	"ebb"		"cad" "bad"	"ace" "bee"	
"a"	"b"	"c"	"d"	"e"	. . .

Radix sort (cont.)

- copy the words from the bucket list back to the list, preserving order
- results in a list with words sorted by last letter

"baa"	"ebb"		"cad"	"ace"	
			"bad"	"bee"	
"a"	"b"	"c"	"d"	"e"	. . .

"baa"	"ebb"	"cad"	"bad"	"ace"	"bee"

- repeat, but now place words into buckets based on next-to-last letter
- results in a list with words sorted by last two letters

"baa"	"ebb"	"ace"		"bee"	
"cad"					
"bad"					
"a"	"b"	"c"	"d"	"e"	. . .

"baa"	"cad"	"bad"	"ebb"	"ace"	"bee"

- repeat, but now place words into buckets based on first letter
- results in a sorted list

"ace"	"baa"	"cad"		"ebb"	
	"bad"				
	"bee"				
"a"	"b"	"c"	"d"	"e"	. . .

"ace"	"baa"	"bad"	"bee"	"cad"	"ebb"

big-Oh analysis?

21

Big-Omega & Big-Theta

Big-Oh represents an asymptotic upper bound on algorithm cost

- but not necessarily a "tight" bound
- if an algorithm is O(N), then it is also O(N^2)

$$f(N) = 5N - 2 < 5N \le 5N^2 \text{ (when } N \ge 1\text{)}$$

to really capture rate of growth, we must prove a tight bound on cost

Big-Omega is an asymptotic lower bound

- an algorithm is Ω(f(N)) if there exists a positive constant C & non-negative integer T such that for all N ≥ T, # of steps required ≥ C*f(N)

Big-Theta is a tight asymptotic bound (both lower and upper)

- an algorithm is θ(f(N)) if it is O(f(N)) and Ω(f(N))

22

Proving a tight bound

to formally prove rate-of-growth, must show Big-Theta

- $f(N) = N^2 + 5N - 2 \leq N^2 + 5N \leq N^2 + 5N^2$ (when $N \geq 1$) $= 6N^2 \Rightarrow O(N^2)$
- $f(N) = N^2 + 5N - 2 \geq N^2 + 5N - 2N$ (when $N \geq 1$) $= N^2 + 3N > 1N^2 \Rightarrow \Omega(N^2)$

$\Rightarrow \theta(N^2)$

as long as we are conservative in proving the upper-bound, the corresponding lower-bound usually follows easily

- so, usually algorithm analysis is stated in terms of Big-Oh (even though Big-Theta is implied)

23

A log is a log

mathematically, $x = \log_b y \leftrightarrow y = b^x$

e.g., $10 = \log_2 1024$, since $1024 = 2^{10}$

properties of logarithms

$\log_b (nm) = \log_b n + \log_b m$

$\log_b (n/m) = \log_b n - \log_b m$

$\log_b (n^r) = r \log_b n$

$\log_a n = \log_b n / \log_b a$

this last property is why we don't care about the log base for Big-Oh

$f(N)$ is $O(\log_a N) \leftrightarrow f(N) <= C \log_a N$ for $N \geq T$

$\leftrightarrow f(N) <= C \log_a N = C (\log_b N / \log_b a) = (C/\log_b a) \log_b N$ for $N \geq T$

$\leftrightarrow f(N)$ is $O(\log_b N)$

24

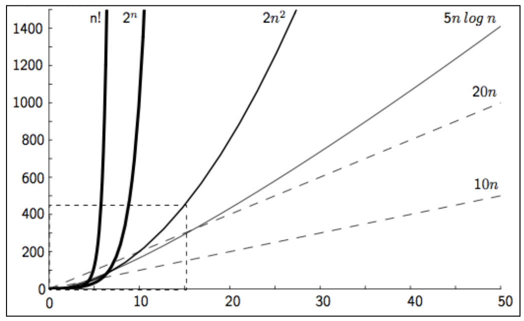
How bad is O(N!)?

recall the first approach to generating anagrams O(L! x D)

Stirling's formula: $n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\epsilon(n)}$ where $\frac{1}{12n+1} \le \epsilon(n) \le \frac{1}{12n}$.

- as n gets large, ε(n) approaches 0, so $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$

➔ O(N!) ~ O(N^N)



25